



Independence Number, Connectivity and Fractional (g, f)-Factors in Graphs

Qiuju Bian^a, Sizhong Zhou^b

^a*School of Mathematics, Shandong University of Technology, Zhangzhou Road 12, Zibo, Shandong 255049, P. R. China*

^b*School of Mathematics and Physics, Jiangsu University of Science and Technology, Mengxi Road 2, Zhenjiang, Jiangsu 212003, P. R. China*

Abstract. Let G be a graph, and let g and f be two integer-valued functions defined on $V(G)$ satisfying $a \leq g(x) \leq f(x) - r \leq b - r$ for any $x \in V(G)$, where a, b and r be three nonnegative integers with $1 \leq a \leq b - r$. In this paper, we verify that G contains a fractional (g, f) -factor if its connectivity $\kappa(G)$ and independence number $\alpha(G)$ satisfy $\kappa(G) \geq \max \left\{ \frac{(b+1)(b-r+1)}{2}, \frac{(b-r+1)^2 \alpha(G)}{4(a+r)} \right\}$. The result is best possible in some sense.

1. Introduction

It is well known that fractional factor problem has wide-range applications in areas such as network design, scheduling and combinatorial polyhedra. For motivation and background to this work, we refer the readers to [12].

In this paper, we consider only finite undirected graphs without loops or multiple edges. Let G be a graph. We denote by $V(G)$ and $E(G)$ its vertex set and edge set. For any $x \in V(G)$, we use $d_G(x)$ to denote the degree of x in G and $N_G(x)$ to denote the set of vertices adjacent to x in G and write $N_G[x] = N_G(x) \cup \{x\}$. For any $S \subseteq V(G)$, we write $N_G(S) = \bigcup_{x \in S} N_G(x)$, use $G[S]$ to denote the subgraph of G induced by S and $G - S = G[V(G) \setminus S]$. A vertex subset S of G is called independent if $G[S]$ has no edges. Given two disjoint subsets A, B of $V(G)$, we write $e_G(A, B)$ for the number of edges in G joining a vertex in A to that in B . We use $\delta(G)$, $\alpha(G)$ and $\kappa(G)$ to denote the minimum degree, the independence number and the connectivity of G , respectively.

Let g and f be two nonnegative integer-valued functions defined on $V(G)$ such that $g(x) \leq f(x)$ for every $x \in V(G)$. A spanning subgraph F of G satisfying $g(x) \leq d_F(x) \leq f(x)$ for every $x \in V(G)$ is a (g, f) -factor of G . Let $h : E(G) \rightarrow [0, 1]$ be a function defined on $E(G)$. If $g(x) \leq \sum_{e \ni x} h(e) \leq f(x)$ holds for every $x \in V(G)$, then we call $G[F_h]$ a fractional (g, f) -factor of G with indicator function h , where $F_h = \{e : e \in E(G), h(e) > 0\}$. A fractional (f, f) -factor is called simply a fractional f -factor. If $f(x) = k$, then a fractional f -factor is called a fractional k -factor.

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Corresponding author: Sizhong Zhou.

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Email addresses: bianqiuju@sdut.edu.cn (Qiuju Bian), zsz_cumt@163.com (Sizhong Zhou)

Many authors have investigated graph factors [1–6, 10, 11, 14] and fractional factors [7–9, 12, 13, 15], but only few results are obtained for the existence of graph factors or fractional factors involving the independent number and the connectivity. In this paper, we study the relationship between the independent number, the connectivity and the fractional (g, f) -factors in graphs, and obtain an independent number and connectivity condition for the existence of the fractional (g, f) -factors in graphs. Our main result will be given in the following section.

2. Main Result

We first show the main result in this paper.

Theorem 2.1. *Let G be a graph, and let a, b and r be three nonnegative integers satisfying $1 \leq a \leq b - r$, and let g, f be two integer-valued functions defined on $V(G)$ with $a \leq g(x) \leq f(x) - r \leq b - r$ for every $x \in V(G)$. If*

$$\kappa(G) \geq \max \left\{ \frac{(b+1)(b-r+1)}{2}, \frac{(b-r+1)^2 \alpha(G)}{4(a+r)} \right\},$$

then G contains a fractional (g, f) -factor.

If $r = 0$ in Theorem 2.1, then we obtain the following corollary.

Corollary 2.2. *Let G be a graph, and let a, b be two integers satisfying $1 \leq a \leq b$, and let g, f be two integer-valued functions defined on $V(G)$ with $a \leq g(x) \leq f(x) \leq b$ for every $x \in V(G)$. If*

$$\kappa(G) \geq \max \left\{ \frac{(b+1)^2}{2}, \frac{(b+1)^2 \alpha(G)}{4a} \right\},$$

then G admits a fractional (g, f) -factor.

If $g(x) \equiv f(x)$ in Corollary 2.2, then we get the following corollary.

Corollary 2.3. *Let G be a graph, and let a, b be two integers satisfying $1 \leq a \leq b$, and let f be an integer-valued function defined on $V(G)$ with $a \leq f(x) \leq b$ for every $x \in V(G)$. If*

$$\kappa(G) \geq \max \left\{ \frac{(b+1)^2}{2}, \frac{(b+1)^2 \alpha(G)}{4a} \right\},$$

then G have a fractional f -factor.

If $a = b = k$ in Corollary 2.3, then we have the following corollary.

Corollary 2.4. *Let G be a graph, and let k be an integer with $k \geq 1$. If*

$$\kappa(G) \geq \max \left\{ \frac{(k+1)^2}{2}, \frac{(k+1)^2 \alpha(G)}{4k} \right\},$$

then G have a fractional k -factor.

3. The Proof of Theorem 2.1

We first show a necessary and sufficient condition for a graph to have a fractional (g, f) -factor obtained by Liu and Zhang [7], which plays an important role in the proof of Theorem 2.1.

Theorem 3.1. ([7]). *Let G be a graph. Then G has a fractional (g, f) -factor if and only if for every subset S of $V(G)$,*

$$\delta_G(S, T) = f(S) + d_{G-S}(T) - g(T) \geq 0,$$

where $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) \leq g(x)\}$.

Proof of Theorem 2.1. We prove Theorem 2.1 by contradiction. Suppose that G satisfies the assumption of Theorem 2.1, but it has no fractional (g, f) -factor. Then using Theorem 3.1, there exists some subset S of $V(G)$ satisfying

$$\delta_G(S, T) = f(S) + d_{G-S}(T) - g(T) \leq -1, \tag{1}$$

where $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) \leq g(x)\}$. Obviously, $T \neq \emptyset$ by (1).

We present the following partition of T : we choose $x_1 \in T$ with $d_{G[T]}(x_1) = \delta(G[T])$. Let $D_1 = N_G[x_1] \cap T$ and $T_1 = T$. If $T - \bigcup_{1 \leq j < i} D_j \neq \emptyset$ for $i \geq 2$, then we write $T_i = T - \bigcup_{1 \leq j < i} D_j$. In the following, we take $x_i \in T_i$ with $d_{G[T_i]}(x_i) = \delta(G[T_i])$ and $D_i = N_G[x_i] \cap T_i$. We continue these procedures until we reach the situation in which $T_i = \emptyset$ for some i , say for $i = s + 1$. It follows from the above definition that $\{x_1, x_2, \dots, x_s\}$ is an independent set of G .

Note that $T \neq \emptyset$. Hence, we have $s \geq 1$. Set $|D_i| = d_i$, we obtain $|T| = \sum_{1 \leq i \leq s} d_i$. We write $U = V(G) \setminus (S \cup T)$ and $\kappa(G - S) = t$. Now, we verify the following claims.

Claim 1. $s \neq 1$ or $U \neq \emptyset$.

Proof. Assume that $s = 1$ and $U = \emptyset$. Then by (1) and our choice of x_1 , we obtain

$$\begin{aligned} -1 &\geq \delta_G(S, T) = f(S) + d_{G-S}(T) - g(T) \geq (a + r)|S| + d_{G-S}(T) - (b - r)|T| \\ &= (a + r)|S| + d_1(d_1 - 1) - (b - r)d_1 \end{aligned}$$

that is,

$$|S| \leq \frac{-d_1^2 + (b - r + 1)d_1 - 1}{a + r}. \tag{2}$$

In terms of (2), we have

$$\begin{aligned} |V(G)| &= |S| + d_1 \leq \frac{-d_1^2 + (b - r + 1)d_1 - 1}{a + r} + d_1 = \frac{-d_1^2 + (a + b + 1)d_1 - 1}{a + r} \\ &\leq \frac{(a + b + 1)^2 - 4}{4(a + r)} \leq \frac{(b + 1)(b - r + 1)}{2}. \end{aligned}$$

Combining this with the hypothesis of Theorem 2.1, we obtain

$$\frac{(b + 1)(b - r + 1)}{2} \geq |V(G)| > \kappa(G) \geq \frac{(b + 1)(b - r + 1)}{2},$$

which is a contradiction. The proof of Claim 1 is complete. □

Claim 2. $d_{G-S}(T) \geq \sum_{1 \leq i \leq s} d_i(d_i - 1) + \frac{st}{2}$.

Proof. According to the choice of x_i , we have

$$\sum_{1 \leq i \leq s} \left(\sum_{x \in D_i} d_{G[T_i]}(x) \right) \geq \sum_{1 \leq i \leq s} d_i(d_i - 1). \tag{3}$$

For the left-hand side of (3), an edge joining $x \in D_i$ and $y \in D_j$ ($i < j$) is counted only once, that is to say, it is counted in $d_{G[T_i]}(x)$ but not in $d_{G[T_j]}(y)$. Thus, we obtain

$$d_{G-S}(T) \geq \sum_{1 \leq i \leq s} d_i(d_i - 1) + \sum_{1 \leq i < j \leq s} e_G(D_i, D_j) + e_G(T, U). \tag{4}$$

According to $\kappa(G - S) = t$ and Claim 1, we have

$$e_G(D_i, \bigcup_{j \neq i} D_j) + e_G(D_i, U) \geq t \tag{5}$$

for all D_i ($1 \leq i \leq s$). It follows from (5) that

$$\sum_{1 \leq i \leq s} (e_G(D_i, \bigcup_{j \neq i} D_j) + e_G(D_i, U)) = 2 \sum_{1 \leq i < j \leq s} e_G(D_i, D_j) + e_G(T, U) \geq st,$$

which implies

$$\sum_{1 \leq i < j \leq s} e_G(D_i, D_j) + e_G(T, U) \geq \frac{st}{2}.$$

Combining this with (4), we obtain

$$d_{G-S}(T) \geq \sum_{1 \leq i \leq s} d_i(d_i - 1) + \frac{st}{2}.$$

This completes the proof of Claim 2. □

It is easy to see that $d_i^2 - (b - r + 1)d_i \geq -\frac{(b - r + 1)^2}{4}$. Combining this with $|T| = \sum_{1 \leq i \leq s} d_i$ and Claim 2, we have

$$\begin{aligned} \delta_G(S, T) &= f(S) + d_{G-S}(T) - g(T) \geq (a + r)|S| + \sum_{1 \leq i \leq s} d_i(d_i - 1) + \frac{st}{2} - (b - r)|T| \\ &= (a + r)|S| + \sum_{1 \leq i \leq s} d_i(d_i - 1) + \frac{st}{2} - (b - r) \sum_{1 \leq i \leq s} d_i \\ &= (a + r)|S| + \sum_{1 \leq i \leq s} (d_i^2 - (b - r + 1)d_i) + \frac{st}{2} \\ &\geq (a + r)|S| - \frac{(b - r + 1)^2 s}{4} + \frac{st}{2}, \end{aligned}$$

that is,

$$\delta_G(S, T) \geq (a + r)|S| - \frac{(b - r + 1)^2 s}{4} + \frac{st}{2}. \tag{6}$$

Claim 3. $-\frac{(b-r+1)^2}{4} + \frac{t}{2} < 0$.

Proof. If $-\frac{(b-r+1)^2}{4} + \frac{t}{2} \geq 0$, then by (6), $s \geq 1$ and $|S| \geq 0$, we obtain

$$\delta_G(S, T) \geq (a + r)|S| - \frac{(b - r + 1)^2 s}{4} + \frac{st}{2} \geq 0,$$

which contradicts (1). The proof of Claim 3 is complete. □

Note that $\alpha(G) \geq \alpha(G[T]) \geq s$ and $\kappa(G) \leq |S| + \kappa(G - S) = |S| + t$. Combining these with (1), (6), Claim 3 and the condition $\kappa(G) \geq \max \left\{ \frac{(b + 1)(b - r + 1)}{2}, \frac{(b - r + 1)^2 \alpha(G)}{4(a + r)} \right\}$ of Theorem 2.1, we have

$$\begin{aligned} -1 &\geq \delta_G(S, T) \geq (a + r)|S| - \frac{(b - r + 1)^2 s}{4} + \frac{st}{2} \\ &= (a + r)|S| + \left(-\frac{(b - r + 1)^2}{4} + \frac{t}{2}\right)s \\ &\geq (a + r)(\kappa(G) - t) + \left(-\frac{(b - r + 1)^2}{4} + \frac{t}{2}\right)\alpha(G) \\ &\geq (a + r)(\kappa(G) - t) + \left(-\frac{(b - r + 1)^2}{4} + \frac{t}{2}\right) \cdot \frac{4(a + r)\kappa(G)}{(b - r + 1)^2} \\ &= (a + r)t \left(\frac{2\kappa(G)}{(b - r + 1)^2} - 1 \right) \geq 0, \end{aligned}$$

which is a contradiction. This completes the proof of Theorem 2.1. □

4. Remark

We show that the condition

$$\kappa(G) \geq \frac{(b-r+1)^2 \alpha(G)}{4(a+r)} = \frac{\left(\frac{b-r+1}{2}\right)^2 \alpha(G)}{a+r}$$

in Theorem 2.1 is sharp by constructing a graph $G = K_{\frac{\left(\frac{b-r+1}{2}\right)^2 s-1}{a+r}} \vee (sK_{\frac{b-r+1}{2}})$, where a, b, r are three nonnegative integers with $1 \leq a = b - r$, s is a sufficiently large integer, $\frac{b-r+1}{2}$ and $\frac{\left(\frac{b-r+1}{2}\right)^2 s-1}{a+r}$ are two integers. It is easy to see that $\alpha(G) = s$ and $\kappa(G) = \frac{\left(\frac{b-r+1}{2}\right)^2 s-1}{a+r} = \frac{\left(\frac{b-r+1}{2}\right)^2 \alpha(G)-1}{a+r}$. Let g and f be two functions defined on $V(G)$ with $g(x) \equiv a$ and $f(x) \equiv b$. In the following, we prove that G has no fractional (g, f) -factor.

We take $S = V\left(K_{\frac{\left(\frac{b-r+1}{2}\right)^2 s-1}{a+r}}\right)$ and $T = V\left(sK_{\frac{b-r+1}{2}}\right)$. Note that $a = b - r$. Thus, we have

$$\begin{aligned} \delta_G(S, T) &= f(S) + d_{G-S}(T) - g(T) = b|S| + d_{G-S}(T) - a|T| \\ &= b \cdot \frac{\left(\frac{b-r+1}{2}\right)^2 s-1}{a+r} + \frac{(b-r+1)s}{2} \cdot \left(\frac{b-r+1}{2} - 1\right) - a \cdot \frac{(b-r+1)s}{2} \\ &= \left(\frac{b-r+1}{2}\right)^2 s-1 - \frac{(b-r+1)s}{2} \cdot \left(b-r - \frac{b-r+1}{2} + 1\right) \\ &= -1 < 0. \end{aligned}$$

Then using Theorem 3.1, G has no fractional (g, f) -factor.

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